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The existence of positive solutions for some nonlinear boundary value problems with linear mixed boundary conditions ☆

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Received 7 June 2004

Available online 16 June 2005

Submitted by A.C. Peterson

Abstract

In this paper we study the existence of positive solutions of the equation

$$\big(\varphi(x')\big)' + a(t)f\big(x(t)\big) = 0,$$

where $\varphi : R \to R$ is an increasing homeomorphism and homomorphism and $\varphi(0) = 0$, subject to linear mixed boundary conditions by a simple application of a fixed point index theorem in cones.
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Keywords: Boundary value problems; Positive solutions; Fixed point index theorem; Cone

1. Introduction

In [1], Erbe and Wang considered the boundary value problem

$$\begin{cases} u'' + a(t)f(u) = 0, \\ \alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

☆ Project supported by CNSF and Foundation of Major Project of Science and Technology of Chinese Education Ministry NSF of Education Committee of Jiangsu Province.

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doi:10.1016/j.jmaa.2004.09.036

they obtained the existence of positive solutions under $f(u)$ satisfying superlinear or sublinear condition.

In [2], Erbe showed there were at least two positive solutions of the equations $-u'' = f(t,u)$, subject to linear boundary conditions, if $f(t,u)$ was superlinear at one end (zero or infinity) and sublinear at the other end.

In [3], Wang studied the existence of positive solution of the equation $(g(u'))' + a(t)f(u) = 0$, where $g(v) = |v|^{p-2}v$, $p > 1$, subject to nonlinear boundary conditions, where $f(u)$ was superlinear or sublinear or superlinear at one end and sublinear at the other end.

In [6], Li discussed the existence of positive solutions for singular boundary value problems with p-Laplacian operators, where $f_0, f_\infty \notin \{0, \infty\}$.

Motivated by the results mentioned above, in this paper we study the existence of positive solutions of quasilinear differential equation.

A projection $\varphi : R \to R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:

(i) if $x \leqslant y$, then $\varphi(x) \leqslant \varphi(y)$, $\forall x, y \in R$;
(ii) φ is a continuous bijection and its inverse mapping is also continuous;
(iii) $\varphi(xy) = \varphi(x)\varphi(y)$, $\forall x, y \in R$.

In this paper, we suppose that $K = \{x \in C[0,1]\colon x(t)$ is nonnegative concave function$\}$, then K is a cone in a Banach space $C[0,1]$.

We will assume that the following conditions are satisfied throughout this paper:

(A) $\varphi : R \to R$ is an increasing homeomorphism and homomorphism;
(B) $f \in C([0,\infty),[0,\infty))$;
(C) $\beta, \delta \in R$, $\beta \geqslant 0$ and $\delta \geqslant 0$;
(D) $a(t) \in C((0,1),[0,\infty))$ and $a(t)$ can be singular at $t = 0$ and $t = 1$.

Let

(E) $0 < \int_0^{1/2} \varphi^{-1}(\int_s^{1/2} a(s_1)\,ds_1)\,ds + \int_{1/2}^1 \varphi^{-1}(\int_{1/2}^s a(s_1)\,ds_1)\,ds < \infty$, where $\varphi^{-1}(x)$ is the inverse function to $\varphi(x)$.

Now we define an operator $T : K \to K$ by

$$(Tx)(t) := \begin{cases} \beta\varphi^{-1}(\int_0^\tau a(s)f(x(s))\,ds) + \int_0^t \varphi^{-1}(\int_s^\tau a(s_1)f(x(s_1))\,ds_1)\,ds, \\ \quad 0 \leqslant t \leqslant \tau, \\ \delta\varphi^{-1}(\int_\tau^1 a(s)f(x(s))\,ds) + \int_t^1 \varphi^{-1}(\int_\tau^s a(s_1)f(x(s_1))\,ds_1)\,ds, \\ \quad \tau \leqslant t \leqslant 1, \end{cases} \tag{1.1}$$

for each $x \in K$. Here $\tau = 0$, if $(Tx)'(0) = 0$; $\tau = 1$, if $(Tx)'(1) = 0$; otherwise, τ is a solution of the equation

$$g_1(t) = g_2(t), \tag{1.2}$$

where

$$g_1(t) := \beta\varphi^{-1}\left(\int_0^t a(s) f\big(x(s)\big)\, ds\right) + \int_0^t \varphi^{-1}\left(\int_s^t a(s_1) f\big(x(s_1)\big)\, ds_1\right) ds,$$
$$0 \leqslant t < 1,$$
$$g_2(t) := \delta\varphi^{-1}\left(\int_t^1 a(s) f\big(x(s)\big)\, ds\right) + \int_t^1 \varphi^{-1}\left(\int_t^s a(s_1) f\big(x(s_1)\big)\, ds_1\right) ds,$$
$$0 < t \leqslant 1.$$

Equation (1.2) has at least one solution in $(0,1)$, because $g_1(t)$ is a nondecreasing continuous function defined on $[0,1)$ with $g_1(0) = 0$ and $g_2(t)$ is a nonincreasing continuous function defined on $(0,1]$ with $g_2(1) = 0$. Moreover, if $\sigma_1, \sigma_2 \in [0,1]$, $\sigma_1 < \sigma_2$, are solutions of (1.2), then we have $a(s) f(x(s)) \equiv 0$ on $[\sigma_1, \sigma_2]$. Therefore, the operator T is well defined.

The purpose of this paper is to study the existence of positive solutions of quasilinear differential equation

$$\begin{cases} (\varphi(x'))' + a(t) f(x(t)) = 0, & t \in (0,1), \\ x(0) - \beta x'(0) = 0, & x(1) + \delta x'(1) = 0, \end{cases} \tag{1.3}$$

subject to linear mixed boundary conditions by a simple application of a fixed point index theorem in cones, where $\varphi : R \to R$ is an increasing homeomorphism and homomorphism and $\varphi(0) = 0$.

It is well known that problem (1.3) is equivalent to the fixed point equation $x(t) = (Tx)(t)$ in $t \in [0,1]$. From the definition of T, we deduce that for each $x \in K$, $(Tx)(t) \in K$, $(Tx)(t)$ satisfies mixed boundary conditions of problem (1.3) and

$$(Tx)(\tau) = \max_{t \in [0,1]} (Tx)(t),$$

since

$$(Tx)'(t) = \begin{cases} \varphi^{-1}(\int_t^\tau a(s) f(x(s))\, ds) \geqslant 0, & 0 < t \leqslant \tau, \\ -\varphi^{-1}(\int_\tau^t a(s) f(x(s))\, ds) \leqslant 0, & \tau \leqslant t < 1, \end{cases}$$

is continuous and nonincreasing in $(0,1)$ and $(Tx)'(\tau) = 0$. Moreover $(\varphi((Tx)'(t)))' = -a(t) f(x(t))$, a.e. $t \in (0,1)$. This shows that $T(K) \subset K$. Since f is continuous, so $T : K \to K$ is completely continuous and each fixed point of T in K is a solution of problem (1.3).

In recent years, there was much attention focused on the existence of positive solutions for p-Laplacian operators (see [3–6,9]), but for an increasing homeomorphism and homomorphism, few works were done as far as we know. We get the existence of multiple positive solutions of problem (1.3) under f satisfying some conditions. Our thoughts come from [7,8].

2. The preliminary lemmas

In this section, we present some lemmas that are important to our main results.

Lemma 2.1 [10]. *Let X be a Banach space, $K \subseteq X$ be a cone in X and $\Omega_r = \{x \in K: \|x\| \leqslant r\}$. Assume that $T : \Omega_r \to K$ is a completely continuous operator, and for $\forall x \in \partial\Omega_r$, $Tx \neq x$. Then we have*

(i) *If $\|x\| < \|Tx\|$, $\forall x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 0$;*
(ii) *If $\|x\| > \|Tx\|$, $\forall x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 1$.*

Lemma 2.2 [3]. *Let* (E) *hold. Then there exists constant $\mu \in (0, 1/2)$ satisfying*

$$0 < \int_{\mu}^{1-\mu} a(t)\,dt < +\infty. \tag{2.1}$$

Furthermore, the function

$$A(t) = \int_{\mu}^{t} \varphi^{-1}\left(\int_{s}^{t} a(s_1)\,ds_1\right) ds + \int_{t}^{1-\mu} \varphi^{-1}\left(\int_{t}^{s} a(s_1)\,ds_1\right) ds$$

is continuous and positive on $[\mu, 1-\mu]$, its minimum value is L and $L > 0$.

Lemma 2.3 [6]. *Let $x \in K$, assume that μ satisfies* (2.1)*. Then $x(t) \geqslant \mu\|x\|$, $t \in [\mu, 1-\mu]$, where $\|x\| = \sup\{x(t):\ 0 \leqslant t \leqslant 1\}$.*

Let

$$\lambda_1 = \frac{1}{(\beta+1)\varphi^{-1}(\int_0^1 a(s)\,ds)} \quad \text{and} \quad \lambda_2 = \frac{4}{\mu^2 L}.$$

Lemma 2.4. *Let* (E) *hold. Assume that*

(E_1) *There exist constants $r > 0$ and $M_1 \in [0, \lambda_1)$ such that $f(x) \leqslant \varphi(M_1 r)$, for $0 \leqslant x \leqslant r$;*
(E_2) *There exist constants $R > 0$ and $M_2 > 2/(\mu L)$ such that $f(x) \geqslant \varphi(M_2 R)$, for $x \in [\mu R, R]$.*

Then problem (1.3) *has at least one positive solution x and $0 < r < \|x\| < R$ or $0 < R < \|x\| < r$.*

Proof. Now we suppose $r < R$. Then if $x \in K$ and $\|x\| = r$. By (E_1), we have

$$\begin{aligned}\|Tx\| = (Tx)(\tau) &\leqslant \beta\varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\,ds\right) + \varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\,ds\right)\\ &\leqslant (\beta+1)\varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\,ds\right)\end{aligned}$$

$$
\begin{aligned}
&\leqslant (\beta+1)\varphi^{-1}\left(\int_0^1 a(s)\varphi(M_1 r)\,ds\right)\\
&\leqslant M_1 r(\beta+1)\varphi^{-1}\left(\int_0^1 a(s)\,ds\right)\\
&= M_1 r\frac{1}{\lambda_1} < r = \|x\|.
\end{aligned}
\tag{2.2}
$$

Let $\Omega_r = \{x \in K\colon\ \|x\| \leqslant r\}$, then (2.2) shows that

$$\|Tx\| < \|x\|, \quad \forall x \in \partial\Omega_r.$$

By the second part of Lemma 2.1, it follows that $i(T, \Omega_r, K) = 1$.

Next let $\Omega_R = \{x \in K\colon\ \|x\| \leqslant R\}$. Then if $x \in K$ and $\|x\| = R$, by Lemma 2.3 and (E_2), we have

$$
\begin{aligned}
2\|Tx\| = 2(Tx)(\tau) &\geqslant \int_0^\tau \varphi^{-1}\left(\int_s^\tau a(s_1) f\bigl(x(s_1)\bigr)\,ds_1\right)ds\\
&\quad + \int_\tau^1 \varphi^{-1}\left(\int_\tau^s a(s_1) f\bigl(x(s_1)\bigr)\,ds_1\right)ds\\
&\geqslant \int_\mu^\tau \varphi^{-1}\left(\int_s^\tau a(s_1) f\bigl(x(s_1)\bigr)\,ds_1\right)ds\\
&\quad + \int_\tau^{1-\mu} \varphi^{-1}\left(\int_\tau^s a(s_1) f\bigl(x(s_1)\bigr)\,ds_1\right)ds\\
&\geqslant \mu M_2 R\left[\int_\mu^\tau \varphi^{-1}\left(\int_s^\tau a(s_1)\,ds_1\right)ds\right.\\
&\quad \left. + \int_\tau^{1-\mu} \varphi^{-1}\left(\int_\tau^s a(s_1)\,ds_1\right)ds\right]\\
&= \mu M_2 R A(\tau) \geqslant \mu M_2 R L > 2R = 2\|x\|, \quad \text{if } \tau \in [\mu, 1-\mu];
\end{aligned}
$$

$$
\begin{aligned}
\|Tx\| = (Tx)(\tau) &\geqslant \int_\mu^{1-\mu} \varphi^{-1}\left(\int_s^{1-\mu} a(s_1) f\bigl(x(s_1)\bigr)\,ds_1\right)ds\\
&\geqslant \mu M_2 R \int_\mu^{1-\mu} \varphi^{-1}\left(\int_s^{1-\mu} a(s_1)\,ds_1\right)ds\\
&= \mu M_2 R A(1-\mu) \geqslant \mu M_2 R L > 2R > \|x\|, \quad \text{if } \tau > 1-\mu;
\end{aligned}
$$

$$\|Tx\| = (Tx)(\tau) \geqslant \int_{\tau}^{1} \varphi^{-1}\left(\int_{\tau}^{s} a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds$$
$$\geqslant \int_{\mu}^{1-\mu} \varphi^{-1}\left(\int_{\mu}^{s} a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds$$
$$\geqslant \mu M_2 R A(\mu) \geqslant \mu M_2 R L > \|x\|, \quad \text{if } \tau < \mu.$$

Therefore, in either case, we have $\|Tx\| > \|x\|$, $\forall x \in \partial\Omega_R$. By the first part of Lemma 2.1, it follows that $i(T, \Omega_R, K) = 0$. Thus we have $i(T, \Omega_R \setminus \Omega_r, K) = -1$, this shows that the fixed point x is a positive solution of problem (1.3).

We can deal with the other case $r > R$ in a similar way and therefore we conclude Lemma 2.4 is valid. □

3. The main results

For convenience, we denote

$$f_0 = \lim_{x \to 0} \frac{f(x)}{\varphi(x)}, \qquad f_\infty = \lim_{x \to +\infty} \frac{f(x)}{\varphi(x)}.$$

The main results are following.

Theorem 3.1. *Let* (E) *hold. Assume that*

(i$_1$) $f_0 = 0$ *and* $f_\infty = +\infty$, *or*
(ii$_1$) $f_0 = +\infty$ *and* $f_\infty = 0$.

Then problem (1.3) *has at least one positive solution.*

Proof. Now suppose $f_0 = 0$ and $f_\infty = +\infty$. Since $f_0 = 0$, we choose $R_1 > 0$ so that

$$0 \leqslant f(x) \leqslant \varphi(\varepsilon x), \tag{3.1}$$

whenever $0 \leqslant x \leqslant R_1$, where $\varepsilon > 0$ satisfies

$$\varepsilon(\beta + 1)\varphi^{-1}\left(\int_0^1 a(s)\,ds\right) < 1. \tag{3.2}$$

Thus if $x \in K$ and $\|x\| = R_1$, then from (3.1) and (3.2), we have

$$\|Tx\| = (Tx)(\tau) \leqslant \beta\varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\,ds\right) + \varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\,ds\right)$$
$$\leqslant (\beta + 1)\varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\,ds\right)$$

$$\leqslant (\beta+1)\varphi^{-1}\left(\int_0^1 a(s)\varphi(\varepsilon R_1)\,ds\right)$$
$$\leqslant \varepsilon R_1(\beta+1)\varphi^{-1}\left(\int_0^1 a(s)\,ds\right) < R_1 = \|x\|. \tag{3.3}$$

Let $\Omega_1 = \{x \in K\colon\ \|x\| \leqslant R_1\}$, then (3.3) shows that

$$\|Tx\| < \|x\|, \quad \forall x \in \partial\Omega_1.$$

By Lemma 2.1(ii), it implies that $i(T, \Omega_1, K) = 1$.

Further since $f_\infty = +\infty$, there exists $R_2 > R_1/\mu$ such that $f(x) \geqslant \varphi(Mx)$, whenever $x \geqslant \mu R_2$, where $M > 0$ is chosen such that

$$\mu LM > 2. \tag{3.4}$$

Let $\Omega_2 = \{x \in K\colon\ \|x\| \leqslant R_2\}$, then $\forall x \in \partial\Omega_2$, from Lemma 2.3 and (3.4), we have

$$\begin{aligned}
2\|Tx\| = 2(Tx)(\tau) &\geqslant \int_0^\tau \varphi^{-1}\left(\int_s^\tau a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\quad + \int_\tau^1 \varphi^{-1}\left(\int_\tau^s a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\geqslant \int_\mu^\tau \varphi^{-1}\left(\int_s^\tau a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\quad + \int_\tau^{1-\mu} \varphi^{-1}\left(\int_\tau^s a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\geqslant \mu M R_2\left[\int_\mu^\tau \varphi^{-1}\left(\int_s^\tau a(s_1)\,ds_1\right) ds \right.\\
&\quad \left. + \int_\tau^{1-\mu} \varphi^{-1}\left(\int_\tau^s a(s_1)\,ds_1\right) ds\right] \\
&= \mu M R_2 A(\tau) \geqslant \mu M R_2 L > 2R_2 = 2\|x\|, \quad \text{if } \tau \in [\mu, 1-\mu]; \\
\|Tx\| = (Tx)(\tau) &\geqslant \int_\mu^{1-\mu} \varphi^{-1}\left(\int_s^{1-\mu} a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\geqslant \mu M R_2 \int_\mu^{1-\mu} \varphi^{-1}\left(\int_s^{1-\mu} a(s_1)\,ds_1\right) ds
\end{aligned}$$

$$= \mu M R_2 A(1-\mu) \geqslant \mu M R_2 L > 2R_2 > \|x\|, \quad \text{if } \tau > 1-\mu;$$

$$\begin{aligned}
\|Tx\| = (Tx)(\tau) &\geqslant \int_\tau^1 \varphi^{-1}\left(\int_\tau^s a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\geqslant \int_\mu^{1-\mu} \varphi^{-1}\left(\int_\mu^s a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\geqslant \mu M R_2 A(\mu) \geqslant \mu M R_2 L > \|x\|, \quad \text{if } \tau < \mu,
\end{aligned}$$

i.e., $\|Tx\| > \|x\|$, $\forall x \in \partial\Omega_2$. By Lemma 2.1(i), it implies that $i(T, \Omega_2, K) = 0$. Thus we have $i(T, \Omega_2 \setminus \Omega_1, K) = -1$, it follows that T has a fixed point $x \in \Omega_2 \setminus \Omega_1$. Furthermore since $0 < R_1 < \|x\| < R_2$, it shows $x(t) > 0$ for $t \in (0,1)$, this shows that the fixed point x is a positive solution of problem (1.3).

Next suppose that $f_0 = +\infty$ and $f_\infty = 0$. Since $f_0 = +\infty$, we choose $R_1 > 0$ such that $f(x) \geqslant \varphi(Mx)$, whenever $0 \leqslant x \leqslant R_1$, where $M > 0$ satisfies (3.4). Let $\Omega_1 = \{x \in K\colon \|x\| \leqslant R_1\}$, then $\forall x \in \partial\Omega_1$, we have

$$\begin{aligned}
2\|Tx\| = 2(Tx)(\tau) &\geqslant \int_\mu^\tau \varphi^{-1}\left(\int_s^\tau a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\quad + \int_\tau^{1-\mu} \varphi^{-1}\left(\int_\tau^s a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\geqslant \mu M R_1 \left[\int_\mu^\tau \varphi^{-1}\left(\int_s^\tau a(s_1)\,ds_1\right) ds \right.\\
&\quad \left. + \int_\tau^{1-\mu} \varphi^{-1}\left(\int_\tau^s a(s_1)\,ds_1\right) ds\right] \\
&\geqslant \mu M R_1 A(\tau) \geqslant \mu M R_1 L > 2R_1 = 2\|x\|, \quad \text{if } \tau \in [\mu, 1-\mu];
\end{aligned}$$

$$\begin{aligned}
\|Tx\| = (Tx)(\tau) &\geqslant \int_0^\tau \varphi^{-1}\left(\int_s^\tau a(s_1) f\big(x(s_1)\big)\,ds_1\right) ds \\
&\geqslant \mu M R_1 \int_\mu^{1-\mu} \varphi^{-1}\left(\int_s^{1-\mu} a(s_1)\,ds_1\right) ds \\
&= \mu M R_1 A(1-\mu) \geqslant \mu M R_1 L > 2R_1 > \|x\|, \quad \text{if } \tau > 1-\mu;
\end{aligned}$$

$$\begin{aligned}\|Tx\| = (Tx)(\tau) &\geqslant \int_\tau^1 \varphi^{-1}\left(\int_\tau^s a(s_1) f\big(x(s_1)\big)\, ds_1\right) ds \\ &\geqslant \int_\mu^{1-\mu} \varphi^{-1}\left(\int_\mu^s a(s_1) f\big(x(s_1)\big)\, ds_1\right) ds \\ &\geqslant \mu M R_1 A(\mu) \geqslant \mu M R_1 L > \|x\|, \quad \text{if } \tau < \mu,\end{aligned}$$

i.e., $\|Tx\| > \|x\|$, $\forall x \in \partial\Omega_1$. By Lemma 2.1(i), it implies that $i(T, \Omega_1, K) = 0$.

Since $f_\infty = 0$, there exists $R_0 > 0$ so that $f(x) \leqslant \varphi(\varepsilon x)$, whenever $x \geqslant R_0$, where $\varepsilon > 0$ satisfies (3.2).

If f is unbound, then we choose $R_2 > R_0 + R_1$ so that $f(x) \leqslant f(R_2)$, for $0 \leqslant x \leqslant R_2$. For $x \in K$ and $\|x\| = R_2$, we have

$$\begin{aligned}\|Tx\| = (Tx)(\tau) &\leqslant \beta\varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\, ds\right) + \varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\, ds\right) \\ &\leqslant \beta\varphi^{-1}\left(f(R_2)\int_0^1 a(s)\, ds\right) + \varphi^{-1}\big(f(R_2)\big)\varphi^{-1}\left(\int_0^1 a(s)\, ds\right) \\ &\leqslant (\beta+1)\varepsilon R_2 \varphi^{-1}\left(\int_0^1 a(s)\, ds\right) < R_2 = \|x\|.\end{aligned}$$

If f is bound, say $f(x) \leqslant \varphi(M_1)$, for all $x \geqslant 0$. In this case, let $R_2 > R_1 + M_1/\varepsilon$. Then for $x \in K$ with $\|x\| = R_2$, we have

$$\begin{aligned}\|Tx\| = (Tx)(\tau) &\leqslant \beta\varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\, ds\right) + \varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\, ds\right) \\ &\leqslant (\beta+1) M_1 \varphi^{-1}\left(\int_0^1 a(s)\, ds\right) \\ &\leqslant (\beta+1)\varepsilon R_2 \varphi^{-1}\left(\int_0^1 a(s)\, ds\right) < R_2 = \|x\|.\end{aligned}$$

Therefore, in either case, we may put $\Omega_2 = \{x \in K\colon \|x\| \leqslant R_2\}$ and we have

$$\|Tx\| < \|x\|, \quad \forall x \in \partial\Omega_2.$$

By Lemma 2.1(ii), it implies that $i(T, \Omega_2, K) = 1$. Thus we have $i(T, \Omega_2 \setminus \Omega_1, K) = 1$, this shows problem (1.3) has a positive solution. □

Theorem 3.2. *Let* (E) *hold. Assume that*

(i$_2$) $f_0 = f_\infty = +\infty$;
(ii$_2$) *There exists* $\rho > 0$, *for* $0 \leqslant x \leqslant \rho$, $f(x) < \varphi(\eta\rho)$, *where*

$$\eta = \frac{1}{(\beta+1)\varphi^{-1}(\int_0^1 a(s)\,ds)}.$$

Then problem (1.3) *has at least two positive solutions* x_1 *and* x_2 *and* $0 < \|x_1\| < \rho < \|x_2\|$.

Proof. Since $f_0 = +\infty$, there exists $0 < R_1 < \rho$ such that $f(x) \geqslant \varphi(M_1 x)$, whenever $0 \leqslant x \leqslant R_1$, where $M_1 > 0$ satisfies $\mu L M_1 > 2$. Let $\Omega_1 = \{x \in K\colon \|x\| \leqslant R_1\}$. By the proof of Theorem 3.1, we have $i(T, \Omega_1, K) = 0$.

Further since $f_\infty = +\infty$, there exists $R_2 > \rho/\mu$ such that $f(x) \geqslant \varphi(M_2 x)$, whenever $x \geqslant \mu R_2$, where $M_2 > 0$ is chosen so that $\mu L M_2 > 2$. Let $\Omega_2 = \{x \in K\colon \|x\| \leqslant R_2\}$. By the proof of Theorem 3.1, we have $i(T, \Omega_2, K) = 0$.

Since (ii$_2$), let $\Omega_3 = \{x \in K\colon \|x\| \leqslant \rho\}$. Then for $x \in K$ and $\|x\| = \rho$,

$$\begin{aligned}\|Tx\| = (Tx)(\tau) &\leqslant (\beta+1)\varphi^{-1}\left(\int_0^1 a(s) f\big(x(s)\big)\,ds\right)\\ &< (\beta+1)\eta\rho\varphi^{-1}\left(\int_0^1 a(s)\,ds\right) = \rho = \|x\|.\end{aligned}$$

By Lemma 2.1(ii), it implies that $i(T, \Omega_3, K)$=1. Thus we have

$$i(T, \Omega_3 \setminus \Omega_1, K) = 1 \quad \text{and} \quad i(T, \Omega_2 \setminus \Omega_3, K) = -1,$$

hence problem (1.3) has two positive solutions x_1 and x_2 and $0 \leqslant \|x_1\| \leqslant \rho \leqslant \|x_2\|$. $\square$

Similarly, we can obtain the following

Theorem 3.3. *Let* (E) *hold. Assume that*

(i$_3$) $f_0 = f_\infty = 0$;
(ii$_3$) *There exists* $\rho > 0$, *for* $\mu\rho \leqslant x \leqslant \rho$, $f(x) > \varphi(\lambda\rho)$, *where* $\lambda = 2/(\mu L)$.

Then problem (1.3) *has at least two positive solutions* x_1 *and* x_2 *and* $0 < \|x_1\| < \rho < \|x_2\|$.

Theorem 3.4. *Let* (E) *hold. Assume that*

(i$_4$) $f_0 = \alpha_1 \in [0, \varphi(\lambda_1/2))$;
(ii$_4$) $f_\infty = \beta_1 \in (\varphi(\lambda_2), \infty)$.

Then problem (1.3) *has at least one positive solution.*

Proof. Since $f_0 = \alpha_1 \in [0, \varphi(\lambda_1/2))$, we choose $r > 0$ so that

$$f(x) \leqslant (\alpha_1 + \varepsilon)\varphi(r) = \varphi\left(\frac{\lambda_1}{2}\right)\varphi(r) = \varphi\left(\frac{\lambda_1}{2}r\right),$$

whenever $0 \leqslant x \leqslant r$, where $\varepsilon > 0$ satisfies $\varepsilon = \varphi(\lambda_1/2) - \alpha_1$. Let $M_1 = \lambda_1/2 \in (0, \lambda_1)$, then (E_1) is satisfied.

Since $f_\infty = \beta_1 \in (\varphi(\lambda_2), \infty)$, there exists $R > r > 0$ so that

$$f(x) \geqslant (\beta_1 - \varepsilon)\varphi(x) \geqslant \varphi(\lambda_2)\varphi(\mu R) \geqslant \varphi(\lambda_2 \mu R) = \varphi\left(\frac{4}{\mu^2 L}\mu R\right) = \varphi\left(\frac{4}{\mu L}R\right),$$

whenever $\mu R \leqslant x \leqslant R$, where $\varepsilon > 0$ satisfies $\varepsilon = \beta_1 - \varphi(\lambda_2)$. Let $M_2 = 4/(\mu L) > 2/(\mu L)$, then (E_2) is satisfied. By Lemma 2.4, we can complete the proof of Theorem 3.4. □

Theorem 3.5. *Let* (E) *hold. Assume that*

(i_5) $f_0 = \alpha_2 \in (\varphi(\lambda_2), \infty)$;
(ii_5) $f_\infty = \beta_2 \in [0, \varphi(\lambda_1/2))$.

Then problem (1.3) *has at least one positive solution.*

Proof. Since $f_0 = \alpha_2 \in (\varphi(\lambda_2), \infty)$, we can choose $R > 0$ so that

$$f(x) \geqslant (\alpha_2 - \varepsilon)\varphi(x) = \varphi(\lambda_2)\varphi(x) = \varphi(\lambda_2 x),$$

whenever $0 \leqslant x \leqslant R$, where $\varepsilon = \alpha_2 - \varphi(\lambda_2)$. So $f(x) \geqslant \varphi(\lambda_2 \mu R) = \varphi(4R/(\mu L))$, whenever $\mu R \leqslant x \leqslant R$. Let $M_2 = 4/(\mu L) > 2/(\mu L)$, then (E_2) is satisfied.

Since $f_\infty = \beta_2$, there exists $\rho > 0$ so that

$$f(x) \leqslant (\beta_2 + \varepsilon)\varphi(x) \leqslant \varphi\left(\frac{\lambda_1}{2}\right)\varphi(x) = \varphi\left(\frac{\lambda_1}{2}x\right),$$

whenever $x \geqslant \rho$, where $\varepsilon = \varphi(\lambda_1/2) - \beta_2$.

If f is unbounded, then we choose $r > \max\{R, \rho\}$ so that $f(x) \leqslant f(r)$, whenever $0 \leqslant x \leqslant r$. So $f(x) \leqslant f(r) \leqslant \varphi(\lambda_1 r/2)$, let $M_1 = \lambda_1/2$, then (E_1) is satisfied.

If f is bounded, say $f(x) \leqslant \varphi(M)$, for all $x \geqslant 0$. Let $r > \max\{\rho, 2M/\lambda_1\}$, then $f(x) \leqslant \varphi(\lambda_1 r/2)$, whenever $0 \leqslant x \leqslant r$, let $M_1 = \lambda_1/2 \in [0, \lambda_1)$, then (E_1) is also satisfied. By Lemma 2.4, we can complete the proof of Theorem 3.5. □

Corollary 3.6. *Let* (E), (E_1), (ii_4) *and* (i_5) *hold. Then problem* (1.3) *has at least two positive solutions* x_1 *and* x_2 *and* $0 < \|x_1\| < r < \|x_2\|$.

Proof. By (ii_4) and the proof of Theorem 3.4, there exist $R_1 > r > 0$ and $M_2 > 2/(\mu L)$ such that $f(x) \geqslant \varphi(M_2 R_1)$, whenever $x \in [\mu R_1, R_1]$. By (i_5) and proof of Theorem 3.5, there exist $R_2 < r$ and $M_2^* > 2/(\mu L)$ such that $f(x) \geqslant \varphi(M_2^* R_2)$, whenever $x \in [\mu R_2, R_2]$. By Lemma 2.4, the conclusion holds. □

Corollary 3.7. *Let* (E), (E_2), (i_4) *and* (ii_5) *hold. Then problem* (1.3) *has at least two positive solutions* x_1 *and* x_2 *and* $0 < \|x_1\| < R < \|x_2\|$.

Proof. By the same way as in Corollary 3.6, we can get Corollary 3.7. □

4. For example and remark

As an example we mention the boundary value problem

$$\begin{cases} (\varphi(x'))' + a(t)f(x(t)) = 0, & t \in (0,1), \\ x(0) - \beta x'(0) = 0, & x(1) + \delta x'(1) = 0, \end{cases}$$

where

$$\varphi(x) = \begin{cases} x^3, & x \leqslant 0, \\ x^2, & x > 0, \end{cases}$$

$\beta \geqslant 0$, $\delta \geqslant 0$, $a(t) \in C((0,1),[0,\infty))$ satisfies (E) and f satisfies the conditions of Theorem 3.1. It is clear that $\varphi : R \to R$ is an increasing homeomorphism and homomorphism and $\varphi(0) = 0$. Hence we generalize boundary value problem with p-Laplacian operators and the results [1–4,6] do not apply to the example.

Acknowledgment

We are very grateful to the referee for his/her valuable suggestions.

References

[1] L.H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. 120 (1994) 743–748.

[2] L.H. Erbe, Multiple positive solutions of some boundary value problems, J. Math. Anal. Appl. 184 (1994) 640–648.

[3] J.Y. Wang, The existence of positive solution for the one dimensional p-Laplacian, Proc. Amer. Math. Soc. 125 (1997) 2275–2283.

[4] W. Sun, W. Ge, The existence of positive solutions for a class of nonlinear boundary value problem, Acta Math. Sinica 4 (2001) 577–580.

[5] H. Chen, X. Liu, Y. Guo, W. Ge, Multiple positive solutions of boundary value problems for quasilinear differential equations, Ann. Differential Equations 19 (2003) 256–260.

[6] B. Li, The existence of positive solutions for singular boundary value problems with p-Laplacian operators, Math. Acta Sci. Ser. A 23 (2003) 257–264.

[7] X. Yang, Positive solutions for nonlinear singular boundary value problems, Appl. Math. Comput. 130 (2002) 225–234.

[8] R.P. Agarwal, Positive solutions for nonlinear singular boundary value problems, Appl. Math. Lett. 12 (1999) 115–120.

[9] Y. Guo, W. Ge, Three positive solutions for the one-dimensional p-Laplacian, J. Math. Anal. Appl. 286 (2003) 491–508.

[10] D. Guo, Nonlinear Functional Analysis, Shangdong Science and Technology Press, Jinan, 1985.